Exact Solutions of 1+1 Dimensional Dilaton Gravity Coupled to Matter

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Abstract

A class of integrable models of 1+1 dimensional dilaton gravity coupled to scalar and electromagnetic fields is obtained and explicitly solved. More general models are reduced to 0+1 dimensional Hamiltonian systems, for which two integrable classes (called s-integrable) are found and explicitly solved. As a special case, static spherical solutions of the Einstein gravity coupled to electromagnetic and scalar fields in any real space-time dimension are derived. A generalization of the 'no-hair' theorem is pointed out and the Hamiltonian formulation that enables quantizing s-integrable systems is outlined.

1. The general 1+1 dimensional dilaton gravity [1] recently attracted a good deal of attention in connection with the black hole physics (for a review and references see e.g. [2]-[4]). An important special case of this theory – the 1+3 dimensional Schwarzschild black hole (SBH) – was recently attempted to quantize by using different approaches [5]-[7]. The key property of SBH, which makes the quantization possible, is integrability[1] [5], [7]. In fact, the most general 1+1 dilaton gravity (DG) also has this nice property. Due to the generalized Birkhoff theorem [4], it actually reduces to a finite dimensional constrained system (FDC) [1], and thus can be explicitly quantized following the approach of Ref.[7].

Though quantizing SBH might seem to be most important, in view of their special role in the Einstein gravity theory, the more general models have to be considered. An explicitly integrable system related to string models and not reducible to FDC had been proposed in [10] (CGHS) and was analyzed in detail for the last three years (see e.g. [11], [12] and references therein). A more complex integrable model, related to the Kerr black hole, was recently studied with employing the full machinery of modern methods for quantizing integrable systems (see [13] where references to other papers of the authors may be found).

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[1] For a relevant review of integrable dynamical systems see e.g. [8], [9]. Integrable systems considered here are explicitly integrable, meaning that their classical solutions can be found in terms of elementary functions or reduced to quadratures.

Though such models are integrable, explicit quantum solutions of them are not easy to construct. For this reason, quantizing reduced models corresponding to the static spherically symmetric sector of gravity coupled to scalars is of significant importance [14]. One of the consequences of the famous 'no hair' theorems (for a general formulation and references see [15]) is that the scalar energy - momentum tensor can not be treated as a perturbation on the SBH background. However, one may hope that a perturbative treatment of non static solutions around exact static ones may still be viable.

All this motivates our search for DG models coupled to scalars (DGS) which are completely integrable or, at least, integrable in the 'static sector' (the precise meaning of this term will be defined later, we will call them s-integrable). We first formulate a general DGS theory, briefly review its relation to the gravity theory in any space - time dimension, and present a very simple solution of the most general DG. Our approach is then applied to constructing apparently new integrable and s-integrable models. The Hamiltonian formulation of the 0+1 dimensional models is also outlined. In the s-integrable case this allows one to apply our approach [7] to quantization. As a byproduct, we point out certain generalization of the 'no hair' theorem. Our treatment is completely local (not using boundary conditions, causality, etc.) but the derived exact solutions may provide a good starting point for a global analysis.

2. The general 1+1 dimensional DG treated here[2] is given by the Lagrangian

$$\mathcal{L} = \sqrt{-g}\left[UR + V + Wg^{ij}\phi_i\phi_j + XF_{ij}F^{ij} + Y + Zg^{ij}\psi_i\psi_j\right], \qquad (1)$$

where U, V, W, X, Y, Z are functions of the dilaton field ϕ while Y may in addition depend on the scalar field ψ; g^{ij} is the contravariant metric tensor, R the scalar curvature, F_{ij} the electromagnetic field tensor. In this paper, the lower letter indices denote partial derivatives ($\phi_i = \partial_i\phi$, etc.), except when used in g_{ij} or F_{ij}. The equations of motion may always be written in the metric $ds^2 = -4f(u,v)dudv$. To simplify the first equation we define the function $w(\phi)$, the importance of which will become clear in a moment, $w'/w \equiv W/U'$ (the prime always denotes the derivative of a function depending on one variable, thus $U' \equiv dU/d\phi$, etc.). Omitting the standard derivations (see e.g. [18], [4]), we write all equations in a convenient form ($Y_\phi \equiv \partial_\phi Y$, $Y_\psi \equiv \partial_\psi Y$)

$$fw(U_i/fw)_i = Z\psi_i^2 , \quad (i = u, v) ; \qquad (2)$$

$$U_{uv} + f(V - XF_{ij}F^{ij} + Y) = 0; \qquad (3)$$

$$2W\phi_{uv} + W'\phi_u\phi_v + U'(\log f)_{uv} + f(V' + X'F_{ij}F^{ij} + Y_\phi) = Z'\psi_u\psi_v ; \quad (4)$$

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[2] Many authors consider the models with $W = \text{const}$ (usually $\frac{1}{2}$ or even 0, see e.g. [1], [3], [4], [16]). As will soon become clear, it is important to keep W arbitrary. Note that we call the dilaton gravity (DG) the integrable $U - X$ part of the Lagrangian (1). The general $U - Z$ model, which we call DGS, may be extended to include many scalar fields, nonabelian gauge fields and spinor fields. Such models naturally emerge in the dimensionally reduced Einstein gravity coupled to matter (see e.g. [17]).

$$(XF_{uv}/2f)_i = 0, \quad (i = u, v); \tag{5}$$

$$(Z\psi_u)_v + (Z\psi_v)_u + fY_\psi = 0. \tag{6}$$

The equations (2) and (3) were derived by varying $\mathcal{L}$ in g^{ij}, the third term in eq.(4) is proportional to the curvature $R = f^{-1}(\log f)_{uv}$. Eq.(5) gives the first integral of motion $Q = XF_{uv}/2f$, then $-XF_{ij}F^{ij} = 2Q^2/X$. It follows that $-XF_{ij}F^{ij}$ may be included in the potential V.

An interesting class of DG described by (1) can be derived by a dimensional reduction of the spherically symmetric sector of the d-dimensional Einstein gravity coupled to scalar and electromagnetic fields having the same symmetry. Following Ref.[19] (see also [20]), let us assume that the d-dimensional metric has the form ($d \equiv n + 2$, $\nu \equiv 1/n$)

$$G_{IJ} = \begin{pmatrix} g_{ij} & 0 \\ 0 & e^{-2\nu\phi} g_{\bar{i}\bar{j}} \end{pmatrix},$$

where $g_{\bar{i}\bar{j}}$ is the metric of the n-dimensional sphere (more generally, a maximally symmetric compact space [19]). As the fields $\phi, \psi, g_{ij}, F_{ij}$ depend only on the coordinates of the 1+1 dimensional subspace, we may integrate over the coordinates of the sphere and get the effective Lagrangian in the form (1), where

$$U = e^{-2\phi}, \quad V = 2e^{-2\phi}(\alpha e^{4\nu\phi} + \Lambda - \beta Q^2 e^{4\phi}), \quad W = 4(1-\nu)e^{-2\phi}, \tag{7a}$$

$$Y = -\gamma y(\psi)e^{-2\phi}, \quad Z = -\gamma e^{-2\phi}. \tag{7b}$$

Here the parameters α, β, γ may depend on ν; α is proportional to the curvature of the sphere, β and γ are normalization constants. We have added to the d-dimensional action the cosmological term (by $R \mapsto R + 2\Lambda$) and included into V the electromagnetic term coming from $X = -e^{-2\phi}/\beta$. As was pointed out in [19], the parameter ν may formally assume any real value. In particular, for $\nu = 0$ and $Q^2 = \gamma = 0$ we obtain the CGHS dilaton gravity. However, to get the CGHS model coupled to the scalar field we have to change Z to $Z_0 = -\gamma_0$ (and choose $y(\psi) = 0$).

Many other models (describing black holes, strings, cosmologies) can be written in the form (1). The approach of our paper may also be used for coupling of DG to many scalar fields, nonabelian gauge fields, spinor fields, etc. To compare different models considered in a rather extensive literature on this subject, one has to keep in mind that, classically, different parametrizations of the potentials in (1) and the Weyl transformations of (1) can be used. For example, for positive definite U we may use the representation $U \equiv e^{-2\phi}$ or $U \equiv \phi^2$. For simplicity, we will often use $U \equiv \phi$. In classical theory, it is always possible. In quantum theory, this parametrization is not necessarily equivalent to the exponential one or to $U \equiv \phi^2$, etc. (see e.g [1]). The Weyl transformation $g_{ij} = \Omega(\phi)\bar{g}_{ij}$ is even more dangerous. However, in the classical framework, it can also be used to compare differently looking models. If two Lagrangians can be identified by using Weyl rescaling and a different choice for U in terms of ϕ, they are equivalent.

To find the Weyl transformation of (1) we only need to know that R transforms in the following way [18]

$$R = \Omega^{-1}(\bar{R} - \bar{g}^{ij}\nabla_i\nabla_j \log \Omega),$$

where $\bar{R}$ is the curvature in terms of $\bar{g}_{ij}$ and ∇_i is the corresponding covariant derivative. The transformed Lagrangian $\bar{\mathcal{L}}$ may now be written in the form (1) with U and Z unchanged and $\bar{V} = \Omega V$, $\bar{Y} = \Omega Y$, $\bar{X} = \Omega^{-1}X$ (because $\bar{F}^{ij} = \Omega^2 F^{ij}$). The transformation of W is more complicated,

$$\bar{W} = W + U'(\log \Omega)',$$

but w transforms simply, $\bar{w} = \Omega w$. Note that, by definition, $\bar{f} = \Omega^{-1}f$. Note also that X and V transform differently but $XF_{ij}F^{ij}$ transforms like V. Thus $fw \equiv \tilde{f}$, $V/w \equiv \tilde{V}$, $XF_{ij}F^{ij}/w$ and Y/w are invariant under the Weyl rescaling. Using this fact it is easy to prove that the equations (2), (3), (5) and (6) are invariant. We will later show that Eq.(4) is also invariant.

3. Let us first discuss integrability of the models with $Y = Z = 0$. Using (2), which now is $(U_i/\tilde{f})_i = 0$, one may write (3) in two equivalent forms (recall that the electromagnetic term is included in V)

$$U_{uv}U_i/\tilde{f} + \tilde{V}U_i = 0. \ (i = u, v),$$

Defining a new function $N(U)$ by $N'(U) = \tilde{V}$ we then see that $M \equiv N(U) + U_uU_v/\tilde{f}$ is locally conserved, i.e. $M_u = 0$ and $M_v = 0$. We may rewrite this Weyl invariant quantity in the coordinate invariant form

$$M \equiv N(U) + U_uU_v/\tilde{f} = N(U) - g^{ij}U_iU_j/w . \tag{8}$$

It is not difficult to prove that the most general locally conserved scalar depending on ϕ, ϕ_i and g^{ij} must be an arbitrary function of M. To show this, one may use the equations (2) and (3) to derive $\partial_k F(\phi, g^{ij}\phi_i\phi_j)$, where $k = u, v$.

Using the local conservation of M, it is very easy to solve the equations (2)-(4). From (2) we have

$$\tilde{f}/U_u = b'(v), \quad \tilde{f}/U_v = a'(u), \tag{9}$$

where $a(u)$ and $b(v)$ are arbitrary functions. This equation shows that $U_a = U_b$ and thus U (and ϕ) depend only on one variable $\tau \equiv a(u) + b(v)$. From (8) and from the conservation of M, it now follows that f can be cast into the form

$$f(u,v) = h(\tau)a'(u)b'(v), \quad \tau \equiv a(u) + b(v). \tag{10}$$

This means that all the equations (2)-(6) are in fact ordinary differential equations for the functions of one variable τ (in what follows we denote the derivatives in τ by dots). Then, using (8) and (9), we express h in terms of U (or ϕ), namely, $h = (M - N)w^{-1}$. Now Eq.(9) tells us that $\dot{U} = hw$ and thus U satisfies the equation $\dot{U} = M - N(U)$ that gives τ in terms of U. This completes the solution

of all the equations. The finite roots, $U = U_0$, of the equation $M = N(U)$ define apparent horizons. Near each horizon one can introduce Kruskal-like coordinates and then study the structure of the singularity, as has recently been done [21] for a special class of DG [3].

As will be shown below for a more general case, the equations for the functions of one variable τ can be written in the Hamiltonian form, which is most suitable for quantizing in our approach [7]. Note that there we used a different metric, which is equivalent to the metric used here due to a simple lemma: two metrics,

$$ds^2 = -4f(u,v)dudv\,, \quad ds^2 = -4\alpha(r)dt^2 + 4\beta(r)dr^2,$$

are related by a coordinate transformation $u = u(t,r)$, $v = v(t,r)$ if and only if there exist two functions $a(u)$ and $b(v)$, such that Eq.(10) is satisfied. Moreover, this transformation depends on one arbitrary function $c(r)$, and also $\alpha(r) = -h(2c(r))$, $\beta(r) = a(r)c'^2(r)$. This lemma shows that the metric is static if and only if f can be represented in the form (10). Thus the above direct derivation of (10) from the field equations reproduces the generalized Birkhoff theorem.

Finishing with DG models I have to emphasize that the above results are essentially not new – I simply derived them in a most general and direct way suitable for a generalization. The locally conserved scalar M was introduced in [22] and generalized in several papers (see [2], [20] and references therein). For the d-dimensional SBH, M is proportional to the black hole mass. The canonical mass of SBH introduced in our papers [7] is of course equivalent to (8). Various other definition of the mass are compared in [23]. The solution of the DG equations based on conservation of M, which is the substance of the generalized Birkhoff theorem, was constructed (in a different form) and applied to SBH physics in [4].

Before proceeding to solving DGS models, I would like to mention that the equations of DG may be presented in a very simple form. Defining a new function $\Phi(U)$ by $\Phi'(U) \equiv (M - N(U))^{-1}$, it is easy to see that Eq.(3) implies that Φ satisfies the D'Alembert equation $\Phi_{uv} = 0$. It is not difficult to prove that, conversely, the equations (2), (3) follow from the three equations

$$M_u = 0,\ \ M_v = 0,\ \ \Phi_{uv} = 0, \tag{11}$$

with the above definitions of M and Φ (by the way, Eq.(4) is satisfied for all solutions of (2) and (3)). Though the equations (11) look simple, it is not clear how one can directly apply them to quantizing black holes and other DG models, while the approach of [7], first reducing DG to FDC, gives an explicit construction of the Hilbert space and observables (this yet remains to be demonstrated for the general DG). Nevertheless, a formulation based on the equations (11) might prove to be useful for physics interpretation and applications including, say, nonspherical or nonstatic perturbations.

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[3] Note that the authors have found the singularity structure consistent with the SBH for the DG models having U, V, W given by (7a) with $\Lambda = Q = 0$. This result is quite natural as these models describe the standard spherical black holes in the d-dimensional Einstein gravity.

4. Unfortunately, the scalar Z-term can not be treated as a perturbation of the integrable DG model. Instead, we have to look for explicitly integrable models with nonzero Z. Then we might hope to treat nonintegrable terms as a perturbation (as we will see below, general DGS models must be not integrable). We first consider a generalization of the string-inspired models, for which U, V, W are arbitrary but $Y = 0$ and $Z = Z_0 = -\gamma_0$ is independent of ϕ. As explained above, the X term may be included in V and we omit it. To make the presentation more compact, we choose $U = \phi$ and use the Weyl invariant combinations $\tilde{f}$ and $\tilde{V}$ instead of f and V. Then, by using Eq.(3) we may rewrite Eq.(4) in a simpler Weyl - invariant form (for $Z = Z_0$ the right-hand side is zero):

$$(\log \tilde{f})_{uv} + \tilde{f}\tilde{V}' = Z'\psi_u\psi_v \,. \tag{12}$$

Adding to this the equation (3) multiplied by a constant g_1, we find

$$F_{uv} + \epsilon\, \mathrm{e}^F[(\tilde{V}' + g_1\tilde{V})\mathrm{e}^{-g_1\phi}] = 0 \,, \tag{13}$$

where $F = \log(\tilde{f}\mathrm{e}^{g_1\phi})$ and ϵ is the sign of $\tilde{f}$. Now, if we choose $\tilde{V}$ so that the expression in the square brackets is a constant, $2g_2$, this equation will be the famous Liouville equation that is known to be integrable [8]. The most general potential satisfying this requirement is

$$\tilde{V} = g_3\mathrm{e}^{-g_1\phi} + \frac{g_2}{g_1}\mathrm{e}^{g_1\phi} \tag{14}$$

(for a general U we may simply replace here ϕ by U). The general solution of the Liouville equation is known, and we may write the solution of Eq.(13)

$$fw\, \mathrm{e}^{g_1\phi} = a'(u)b'(v)[1 + g_2a(u)b(v)]^{-2}, \tag{15}$$

where $a(u)$ and $b(v)$ are arbitrary functions and so ϵ can be included in their definition. Using this solution and Eq.(3) with the potential (14), we can also reduce it to the Liouville equation

$$\Phi_{ab} + g_3\mathrm{e}^{-2g_1\Phi} = 0 \,. \tag{16}$$

Here $\Phi \equiv \phi + g_1^{-1}\log(1 + g_2ab)$ is regarded as a function of the new variables a, b. The exact general solution of (16) gives Φ in terms of two new arbitrary functions $A(a)$ and $B(b)$

$$\mathrm{e}^{-2g_1\Phi} = A'(a)B'(b)[1 - g_1g_3A(a)B(b)]^{-2}. \tag{17}$$

Finally, we have to show that the equations (2) and (6) are also satisfied. In fact, these equations define $\psi(u, v)$ satisfying the D'Alembert equation $\psi_{uv} = 0$ (or, equivalently, $\psi_{ab} = 0$). To prove this, it is sufficient to show that $[\tilde{f}(\phi_a/\tilde{f})_a]_b = 0$. This can be checked explicitly by using (13) and (16).

A simpler model corresponds to $g_1 = 0$. As the above expressions are singular in this limit, we define $\bar{g}_3 \equiv g_3 + g_2/g_1$. Then, in the limit $g_1 \to 0$

$$\tilde{V} = \bar{g}_3 + 2g_2\phi\,. \tag{18}$$

To solve the equations with this potential we may simply put $g_1 = 0$ in Eq.(15). Then defining new arbitrary functions $\bar{A}(a)$ and $\bar{B}(b)$ by $A(a) = a + g_1\bar{A}(a)$ and $B(b) = b + g_1\bar{B}(b)$ we find, by collecting the first-order terms in (17),

$$\phi = -\frac{\bar{g}_3}{2g_2} - \bar{A}'(a) - \bar{B}'(b) + 2g_2\frac{\bar{A}(a)b + \bar{B}(b)a}{1 + g_2ab}\,. \tag{19}$$

In the limit $g_1 \to 0$ and $g_2 \to 0$, our model degenerates to the models related to that of [10] by Weyl rescalings and parametrizations[4] of U (with $U = \phi$, this model is defined by $w = \phi$, $\tilde{V} = 4\lambda^2$ and $Z = -\gamma_0$). Its solution can be obtained by an obvious further redefinition of $\bar{A}$ and $\bar{B}$.

5. Though all the above equations are integrable and their general classical solutions have been written explicitly, the full quantum treatment of our integrable system is a separate and yet unsolved problem. The exact solutions of the s-integrable systems introduced below are easier to quantize. In addition, they give us exact spherically symmetric static solutions of the Einstein gravity non trivially coupled to matter in any space-time dimension that may be useful in various applications.

As we have seen, even the simplest DGS models have nonstatic solutions (we call 'static' the solutions effectively depending of one variable τ). Nevertheless, the static solutions form an interesting subclass of the solutions to Eqs.(2)-(6) if we use Eq.(10) as *Ansatz* and suppose that all other fields depend on one variable τ. Then the metric is static, and we will show that the functions depending on τ are coordinates of a constrained Hamiltonian system (the constraint is the Hamiltonian H itself, i.e. $H = 0$). This system is integrable if there exist two more integrals depending on the coordinates h, ϕ, ψ and velocities $\dot{h}$, $\dot{\phi}$, $\dot{\psi}$ (or momenta introduced below) and the system of the first-order equations defined by these integrals is explicitly integrable (recall that Eq.(5) is always integrable). Then we call the DGS model s-integrable. If $Y = 0$, as we assume in this paper, the equation (6) that now is simply $(Z\dot{\psi}) = 0$, gives the integral $C_0 = Z\dot{\psi}$. Thus the problem is to find one more integral.

The remaining ordinary differential equations are ($U \equiv \phi$):

$$hw(\dot{\phi}/hw) \equiv \ddot{\phi} - W\dot{\phi}^2 - \dot{\phi}F = Z\dot{\psi}^2\,; \tag{2*}$$

$$\ddot{\phi} + hV = 0\,; \tag{3*}$$

$$2W\ddot{\phi} + W'\dot{\phi}^2 + \dot{F} + hV' = Z'\dot{\psi}^2\,. \tag{4*}$$

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[4] In the classical framework, they all are equivalent and thus explicitly integrable. However, in general, their quantum versions are not equivalent due to anomalies (see e.g. [12]).

For convenience, we will use the notation $F \equiv \dot{h}/h$ (not to be confused with F above). From (2^*) and (3^*) we immediately derive the integral

$$L \equiv W\dot{\phi}^2 + \dot{\phi}F + hV + Z\dot{\psi}^2 = 0. \tag{20}$$

We will see shortly that L is the Hamiltonian of a FDC system giving all the equations of motion if it is constrained to be zero.

To derive an additional integral, we exclude $\dot{\psi}^2$ from (4^*) by using (2^*)

$$(2WZ - Z')\ddot{\phi} + (WZ)'\dot{\phi}^2 + hV'Z + (FZ)\dot{} = 0. \tag{21}$$

Here we may either replace h by $-\ddot{\phi}/V$ or exclude $\dot{\phi}^2$ by using (20). In both cases we can find potentials for which (21) has an integral.

The first approach allows us to rewrite (21) in the form

$$(A(\phi)\dot{\phi})\dot{} + (ZF)\dot{} = 0, \quad A(\phi) \equiv 2ZW - Z' - ZV'/V,$$

if $A'(\phi) = (ZW)'$. This gives the differential equation $[\log(\tilde{V}Z)]' = g_1 Z^{-1}$, where g_1 is an arbitrary integration constant. Solving this equation we may express Z in terms of $\tilde{V} \equiv V/w$, or $\tilde{V}$ in terms of Z (recall that $N'(\phi) \equiv \tilde{V}$):

$$Z = (g_2 + g_1 N(\phi))\tilde{V}^{-1}, \quad \tilde{V} = g_2 e^{g_1 z(\phi)} Z^{-1}, \tag{22}$$

where $z'(\phi) \equiv Z^{-1}$. With the potential satisfying (22), we have the integral

$$ZF + (ZW - g_1)\dot{\phi} = C_1. \tag{23}$$

The three available integrals C_0, C_1 and L allow us to find the general solution to all the equations. Indeed, using the above equations, it is not difficult to express $\dot{\phi}$ in terms of C_0, C_1 and of $x \equiv hVZ$. Then one can derive the following first - order equations for $\phi(\tau)$ and $x(\tau)$

$$2g_1\dot{\phi} = -C_1 + R(x), \quad \dot{x} = xR(x)/Z(\phi), \tag{24}$$

where we introduced the notation $R = [C_1^2 - 4g_1(C_0^2 + x)]^{\frac{1}{2}}$ (note that the square root may be positive or negative). From these two equations we derive one explicitly integrable equation

$$2g_1 x \frac{dz}{dx} = 1 - C_1 R^{-1}(x). \tag{25}$$

The solution of this equation expresses z as an elementary function of x, C_0, C_2 and of one additional integration constant (let me omit this simple but tedious algebra). To complete integrating the whole system, we have to express ϕ in terms of z using the definition of $z(\phi)$. Then $Z(\phi)$ can be expressed in terms of z, which is a known elementary function of x, and the second equation in (24) is reduced to a quadrature (in general, not elementary).

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The second approach to deriving an additional integral is to exclude from (21) the terms containing the derivatives of ϕ by using (3*) and (20). In this way we obtain the equation (we also replace h by the identical expression $\dot{h}/F$)

$$F[(FZ)\dot{} - FZ(\log WZ)\dot{}] + \dot{h}VZ[(\log V/W)' - 2W] - C_0^2(\log WZ)'(\log h)\dot{} = 0.$$

Dividing this by $\frac{1}{2}W^2Z$ we make the first term the total derivative $(F^2/W^2)\dot{}$. Now, to obtain an integral we choose the potential so as to make the multipliers of $\dot{h}$ and $(\log h)\dot{}$ constant. Solving the corresponding equations for the potentials, we derive the most general potentials for which the above equation is the total derivative of

$$(F/W)^2 + 4\bar{g}_1 h + 2\bar{g}_2 C_0^2 \log h = \bar{C}_1 , \tag{26}$$

which is the integral we are looking for. The potentials satisfying our requirement are now defined by two relations (it must be clear now why it was important to keep w arbitrary):

$$V = W(\bar{g}_4 w^2 - \bar{g}_1), \quad Z^{-1} = W(\bar{g}_3 + \bar{g}_2 \log w), \tag{27}$$

where $\bar{g}_1$ - $\bar{g}_4$ are arbitrary real constants. Using (26) and (20) one can derive the first-order differential equations for h and $\bar{h} \equiv hw^2$

$$\dot{\bar{h}} = \bar{h}WR_1(\bar{h}), \quad \dot{h} = hWR_2(h), \tag{28}$$

where

$$R_1 = (\bar{C}_1 - 4\bar{g}_3 C_0^2 - 4\bar{g}_4\bar{h} - 2\bar{g}_2 C_0^2 \log \bar{h})^{\frac{1}{2}}, \quad R_2 = (\bar{C}_1 - 4\bar{g}_1 h - 2\bar{g}_2 C_0^2 \log h)^{\frac{1}{2}}.$$

Thus, to find the relation between ϕ and h we have to solve the equation

$$\frac{d\bar{h}}{dh} = \frac{\bar{h}R_1(\bar{h})}{hR_2(h)} . \tag{29}$$

Integrating this equation reduces to one non elementary quadrature (it is elementary if $\bar{g}_2 = 0$).

By the way, with $\bar{g}_2 = 0$ we can derive the exact static spherically symmetric solution of the Einstein gravity coupled to the scalar and electromagnetic fields in any space-time dimension (not necessarily integer). The potentials in Eqs.(7) satisfy Eq.(25) if (recall that we must choose the representation $U = \phi$, in which $W = (1 - \nu)/\phi$, $w = \phi^{1-\nu}$, $Z = -\gamma\phi$)

$$2\alpha = \bar{g}_4(1 - \nu), \quad 2\beta Q^2 = \bar{g}_1(1 - \nu), \quad \gamma^{-1} = -\bar{g}_3(1 - \nu), \quad \Lambda = 0. \tag{30}$$

The integral of motion $\bar{C}_1$ is simply

$$\bar{C}_1 = [F^2\phi^2 + 8\beta(1 - \nu)Q^2h](1 - \nu)^{-2}.$$

One can see that $\nu = 1$ corresponding to the space-time dimension $d = 3$ is not allowed (this is also true for DG); our method is applicable to any other ν.

We will not consider the solutions and their physics applications in more detail. This will be done in a forthcoming publication. Our aim was to construct a general framework for applications which may be pursued in different directions. However, I would like to mention two important points. First, the solutions of the equations (25) or (28) expressing h in terms of ϕ have several branches depending on the integrals C. With changing C, new solutions emerge by bifurcations. Second, examining exact solutions of the static equations with the scalar coupling, I have checked for them the 'no hair' theorem. Indeed, all exact solutions do not give a black hole type singularity in the following strong sense: for any fixed values of C_0, C_1 (or $\bar{C}_1$) and any finite value of ϕ the metric function h has no zeroes. In fact, to prove that h has no zeroes, it is sufficient to use Eq.(3*) and the constraint (20) that are valid in general, without integrability (under rather mild restrictions on the potentials and using some local existence theorem for nonlinear differential equations). One may also relax the condition $Y = 0$, consider more general coupling to several scalar fields, etc. Although it would be not very difficult to prove the above statements, this would distract us from the main subject of this paper and these results will be presented in a separate publication.

6. The final subject of this paper is the Hamiltonian formulation of the 'static' equations of DGS. It is not difficult to show that Eqs.$(2^*) - (4^*)$ as well as the constraint (20) and the omitted equation $(Z\dot{\psi})\dot{} = 0$ can be derived by varying the Lagrangian

$$\mathcal{L}^{(s)} \equiv (L - hV)/l - lhV \tag{31}$$

in all variables including the Lagrangian multiplier $l(\tau)$ (Eq.(20) is reproduced in the gauge $l = 1$). Introducing the momenta

$$lp_h = \dot{\phi}/h, \quad lp_\phi = 2W\dot{\phi} + \dot{h}/h, \quad lp_\psi = 2Z\dot{\psi},$$

we find the Hamiltonian $\mathcal{H}^{(s)} = lH^{(s)}$, where

$$H^{(s)} = hp_h p_\phi - Wh^2 p_h^2 + hV + p_\psi^2/4Z. \tag{32}$$

Now one may express the integrals of motion in terms of the canonical coordinates and momenta and check that their Poisson brackets with $H^{(s)}$ vanish when these variables satisfy the canonical equations of motion (including the constraint $H^{(s)} = 0$). Of course, the canonical equations are equivalent to the Lagrangian ones that were solved above (with $l = 1$)[5].

Note that the form of the Lagrangian and Hamiltonian is not uniquely defined due to a freedom in the choice of the Lagrangian multiplier $l(\tau)$. We may multiply $H^{(s)}$ by a function of the coordinates, $\lambda(h, \phi, \psi)$ (not having zeroes inside the domain of definition of the coordinates) and correspondingly divide $l(\tau)$ by λ.

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[5]The necessity of this gauge fixing is related to the fact that we were using the conformally flat metric depending on one function $f(u, v)$. If we would use the metric defined by $\alpha(r)$ and $\beta(r)$, a Lagrangian multiplier will emerge automatically [7].

The new Hamiltonian $\bar{\mathcal{H}}^{(s)} \equiv \bar{l}\bar{H}^{(s)}$ define the same equations of motion due to the constraint $H^{(s)} = 0$. Such a freedom is useful because the new Hamiltonian may have additional integrals of motion. This observation was used in our approach to quantizing black holes [7]. In that case, the mass of the black hole is proportional to M defined in (8). It is conserved when the scalar field is completely decoupled, i.e. for $C_0 = 0$. For the 'static' case, the mass function is simply

$$M = N(\phi) + hp_h^2/w \equiv N(\phi) + \dot{\phi}^2/hw \ . \tag{33}$$

When $p_\psi \equiv 2C_0 \neq 0$, M is not conserved but, for s-integrable models there may exist other integrals of motion [14]. In the s-integrable models, the integrals C_1 and $\bar{C}_1$ found in this paper play the role of M. Thus, it is not difficult to show that

$$C_1 = -g_1 \frac{w}{p_h} \left(M + \frac{g_2}{g_1} + \frac{p_\psi^2}{4g_1 hw} \right) . \tag{34}$$

When $p_\psi \equiv 2C_0 = 0$ the factor w/p_f in (34) becomes an additional integral of motion because $C_2 \equiv p_h/w = \dot{\phi}/hw$ is independent of τ due to Eq.(2*). A relation similar to (34) may be derived for $\bar{C}_1$. We write it only for $p_\psi = 0$:

$$\bar{C}_1 = (M^2 - 4\bar{g}_1\bar{g}_4)/C_2^2 \ .$$

Of course, the Lagrangian and Hamiltonian formulations are valid in general, when additional integrals of motion are not known or even do not exist. For nonlinear Hamiltonian systems with two or more independent coordinates the existence of such an integral is a rare event. Apparently simple systems with two coordinates are not integrable and thus exhibit complex phenomena known as dynamical chaos. A famous example is the Henon - Heiles system of two oscillators with cubic couplings between them (see e.g. [24]). If one compares our general 'static' system (32) with fixed p_ψ to such well-studied nonitegrable systems, one will hardly believe that it can be integrable (integrability of the DGS field theory looks even less probable). This does not mean that further examples of integrable DGS can not be discovered. Of course, our simple approach is not a suitable framework for such a general search while general criteria for integrability are not known. Yet, one may try to compare DGS equations to known classes of integrable systems [8], [9]. Especially interesting are systems related to solitons [25], [26]. On the other hand, chaotic phenomena in classical nonintegrable DGS models might be of significant physics interest. In quantum framework, nonintegrable models may still be useful if they can be treated perturbatively on some explicitly integrable background (like our s-integrable models).

7. In summary, we have studied a general dilaton gravity coupled to electromagnetic and scalar fields described by Eq.(1). For the $U - X$ models (DG) we presented some known results in a more compact and, hopefully, simpler form (e.g. Eq.(11)). For the $U - Z$ models (DGS) we concentrated on a somewhat simpler case $Y = 0$ and obtained a class of integrable theories with the constant

potential Z, thus generalizing the well-known CGHS dilaton gravity (equations (13) - (19)). For DGS with arbitrary Z we have found two classes of s-integrable systems and constructed for them exact 'static' solutions (equations (21)-(29)). A special case of the s-integrable DGS gives new exact solutions of the Einstein gravity coupled to matter in any space-time dimension. We also pointed out a certain generalization of the 'no-hair' theorem. By constructing the Hamiltonian formulation of the s-integrable systems we have paved a way to their quantizing. A detailed discussion of these results and further applications will be presented elsewhere.

Acknowledgments

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